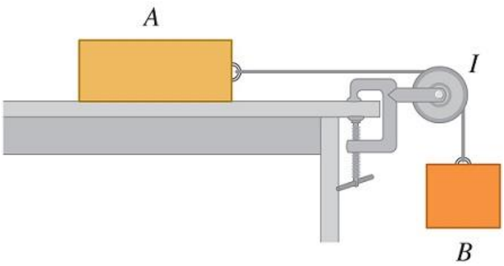
PHYS 101 – General Physics I Final Exam Solution (17.05.2022)

1. The pulley in the figure has radius R and a moment of inertia I. The rope does not slip over the pulley, and the pulley spins on a frictionless axle. The coefficient of kinetic friction between block A and the tabletop is μ_k. The system is released from rest, and block B descends. Block A has mass m_A and block B has mass m_B.



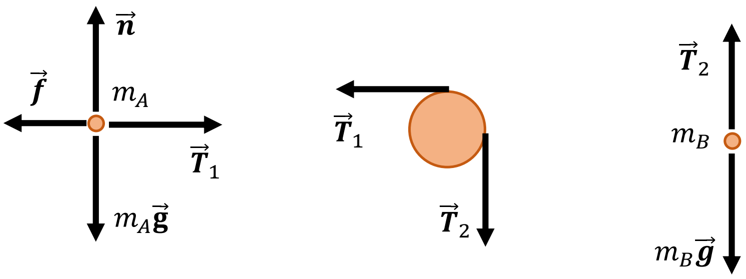
(a) (4 Pts.) Draw free body diagrams for both blocks and the pulley.

(b) (6 Pts.) Find the acceleration of the blocks.

(c) (5 Pts.) Find the expression for the speed of block A as a function of the distance h block B descends.

(d) (5 Pts.) Find the work done on the system by the friction force as a function of the distance h block B descends.

Solution: (a)



(b)

$$T_1 - f = m_A a\,, \qquad RT_2 - RT_1 = I\alpha\,, \qquad m_B g - T_2 = m_B a\,, \qquad n - m_A g = 0\,, \qquad f = \mu_k n\,, \qquad \alpha = \frac{a}{R}$$

$$T_1 - \mu_k m_A g = m_A a\,, \qquad T_2 - T_1 = \frac{I}{R^2}a\,, \qquad m_B g - T_2 = m_B a \quad \rightarrow \quad a = \left(\frac{m_B - \mu_k m_A}{m_B + m_A + I/R^2}\right) g$$

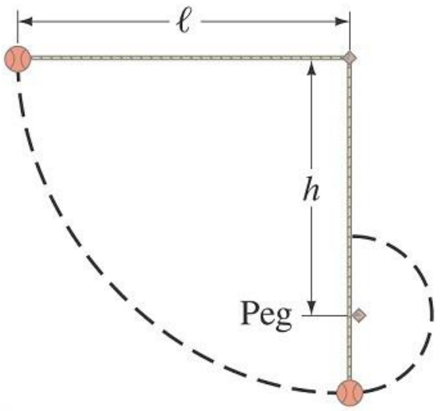
(c)

$$v^2 - v_0^2 = 2ah \quad \rightarrow \quad v = \sqrt{\left(\frac{m_B - \mu_k m_A}{m_B + m_A + I/R^2}\right) 2gh}$$

(d)

$$W_f = -fh \quad \rightarrow \quad W_f = -\mu_k m_A g h$$

2. A ball is attached to a horizontal cord (assumed massless) of length ℓ whose other end is fixed.



(a) (5 Pts.) If the ball is released from rest, what will be its speed at the lowest point of its path?

(b) (15 Pts.) A peg is located a distance h directly below the point of attachment of the cord. What is the minimum value of h if the ball is to swing in a complete circle centered on the peg?

Solution:

(a) Total energy is conserved. Speed v_B of the ball at the bottom is

$$mg\ell = \frac{1}{2}mv_B^2 \quad \rightarrow \quad v_B = \sqrt{2g\ell}\,.$$

(b) The ball will start to move around a circle of radius $\ell - h$. Total energy is conserved. The speed v_T of the ball at the top of its swing is

$$\frac{1}{2}mv_B^2 = \frac{1}{2}mv_T^2 + 2mg(\ell - h) \quad \rightarrow \quad v_T = \sqrt{2g(2h - \ell)}\,.$$

To complete a full swing with the cord going slack exactly at the top, we need to have

$$\frac{mv_T^2}{\ell - h} \geq mg \quad \rightarrow \quad v_T^2 \geq (\ell - h)g \quad \rightarrow \quad 2g(2h - \ell) \geq (\ell - h)g \quad \rightarrow \quad h_{\min} = \frac{3}{5}\ell\,.$$

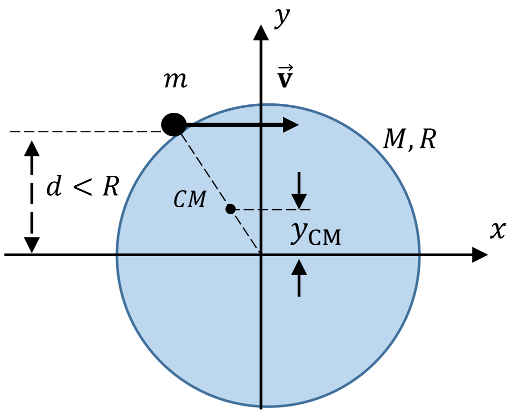
3. A hockey puck is a cylinder of mass M and radius R which has moment of inertia $I = MR^2/2$ around its center. The puck is motionless on ice (which has negligible friction), when a small particle of mass m hits it with velocity $\vec{\mathbf{v}} = v_0\hat{\imath}$. The line of motion of the small particle is a distance d away from the center of the puck as shown in the figure.The small particle sticks to the puck and they move together subsequently.

Solution:

(a) (5 Pts.) Indicate whether the following quantities are conserved or not conserved in this collision.

Quantity	Conserved	Not Conserved
Total Mechanical Energy		X
Momentum in the x direction	X	
Momentum in the y direction	X	
Angular momentum with respect to the collision point	X	
Angular momentum with respect to the center of the puck	X	

b) (5 Pts.) Find the final velocity of the total center of mass of the system.



Linear momentum is conserved.

$$m\vec{\mathbf{v}} = (m + M)\vec{\mathbf{V}}_{\mathrm{CM}} \quad \rightarrow \quad \vec{\mathbf{V}}_{\mathrm{CM}} = \frac{m\vec{\mathbf{v}}}{m + M} = \frac{mv_0\,\hat{\mathbf{i}}}{m + M}.$$

c) (10 Pts.) Find the final angular velocity of the combined puck-particle system.

The puck with the particle embedded at its rim will be rotating around their center of mass, and angular momentum with respect to the center of mass is conserved.

$$y_{\mathrm{CM}} = \frac{m\,d}{m + M}, \qquad \frac{d}{R} = \frac{y_{\mathrm{CM}}}{r} \quad \rightarrow \quad r = \frac{m\,R}{m + M},$$

where r is the radial distance between the center of mass and the origin. Moment of inertia of the puck with the particle embedded at the rim around the center of mass is found by using the paralllel axis theorem.

$$I_{\mathrm{CM}} = \frac{1}{2}MR^2 + M\left(\frac{m\,R}{m + M}\right)^2 + m\left(\frac{M\,R}{m + M}\right)^2 \quad \rightarrow \quad I_{\mathrm{CM}} = \frac{1}{2}\left(\frac{M + 3m}{M + m}\right)MR^2.$$

Conservation of angular momentum with respect to the center of mass means

$$L_i = mv_0(d - y_{\mathrm{CM}}) = \frac{mMv_0 d}{M + m}, \qquad L_f = I_{\mathrm{CM}}\omega \quad \rightarrow \quad \omega = \frac{mMv_0 d}{I_{\mathrm{CM}}(M + m)}.$$

$$\omega = \frac{2mv_0 d}{(M + 3m)R^2}$$

4. Consider a spherical planet with mass M_P which has a radius R_P.

(a) (5 Pts.) Ignoring the effect of all other heavenly bodies around, what is the minimum initial velocity for a projectile that is fired from the surface to go infinitely far away from the planet.

Now assume that this planet has a moon which has mass M_m and radius R_m. The moon is at a circular orbit around the planet and the distance between the centers of the planet and the moon is always L ($L > R_P + R_m$). You can ignore the slow rotation of the moon around the planet and assume that it is stationary.

(b) (10 Pts.) What is the minimum initial speed for a projectile fired from the planet surface so that it can reach the moon?

(c) (5 Pts.) What will be the minimum speed of a projectile that hits the moon if it is fired from the planet's surface?

Solution: (a)

$$\frac{1}{2}mv_e^2 - G\frac{M_P m}{R_p} = 0 \quad \rightarrow \quad v_e = \sqrt{\frac{2GM_p}{R_p}}.$$

(b) To reach the moon, the projectile must reach the point where the attractive forces of the planet and the moon cancel each other. From that point on, the attractive force of the moon will pull the projectile. Let d be the distance of this point to the center of the planet. Then

$$G\frac{M_p m}{d^2} = G\frac{M_m m}{(L-d)^2} \quad \rightarrow \quad d = \frac{L}{1+\sqrt{M_m/M_p}}.$$

Potential energy of the projectile as a function of the distance r measured from the center of the planet is

$$U(r) = -G\frac{M_P m}{r} - G\frac{M_m m}{L-r} = -Gm\left(\frac{M_P}{r} + \frac{M_m}{L-r}\right)$$

To reach the point $r = d$, the minimum initial speed of the projectile on the surface of the planet must satisfy

$$\frac{1}{2}mv_{\min}^2 - Gm\left(\frac{M_P}{R_P} + \frac{M_m}{L-R_P}\right) = -Gm\left(\frac{M_P}{d} + \frac{M_m}{L-d}\right) \quad \rightarrow \quad v_{\min}^2 = 2G\left(\frac{M_P}{R_P} + \frac{M_m}{L-R_P} - \frac{M_P}{d} - \frac{M_m}{L-d}\right)$$

$$v_{\min} = \sqrt{2GM_P\left[\frac{1}{R_P} + \frac{M_m/M_P}{L-R_P} - \frac{1}{L}\left(1+\sqrt{M_m/M_P}\right)^2\right]}$$

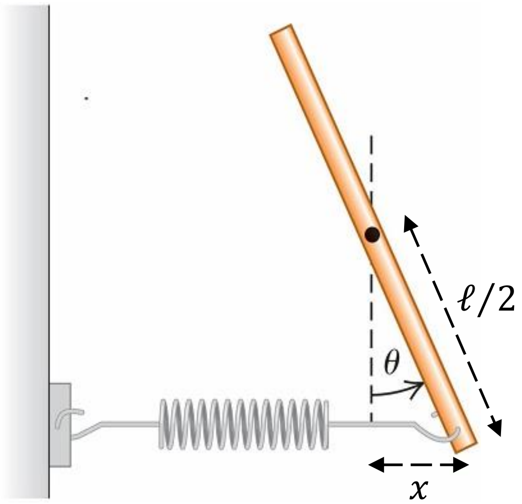
(c) Noting that total energy is conserved, we have

$$-Gm\left(\frac{M_P}{d} + \frac{M_m}{L-d}\right) = \frac{1}{2}mv_{\min}^2 - Gm\left(\frac{M_m}{R_m} + \frac{M_P}{L-R_m}\right) \quad \rightarrow \quad v_{\min}^2 = 2G\left(\frac{M_m}{R_m} + \frac{M_P}{L-R_m} - \frac{M_P}{d} - \frac{M_m}{L-d}\right)$$

$$v_{\min} = \sqrt{2GM_m\left[\frac{1}{R_m} + \frac{M_P/M_m}{L-R_m} - \frac{1}{L}\left(1+\sqrt{M_m/M_P}\right)^2\right]}$$

5. (20 Pts.) A thin, uniform rod with mass M and length ℓ is pivoted without friction about an axis through its midpoint and perpendicular to the rod. A horizontal spring with force constant k is attached to the lower end of the rod, with the other end of the spring attached to a rigid support. The rod is displaced by a small angle θ from the vertical equilibrium position and released. Find the period of small oscillations.

($I_{\text{CM}} = M\ell^2/12$ for the rod.)



Solution:

Method 1.

$$\tau = I\alpha \quad \rightarrow \quad \frac{1}{12}M\ell^2\frac{d^2\theta}{dt^2} = -\frac{\ell}{2}kx\sin\left(\theta + \frac{\pi}{2}\right)$$

For small angles

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta \approx 1\,, \qquad x = \frac{\ell}{2}\sin\theta \approx \frac{\ell}{2}\theta \quad \rightarrow \quad \frac{1}{12}M\ell^2\frac{d^2\theta}{dt^2} = -\frac{\ell^2}{4}k\,\theta \quad \rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{3k}{M}\theta$$

$$\omega = \sqrt{\frac{3k}{M}} \quad \rightarrow \quad T = 2\pi\sqrt{\frac{M}{3k}}$$

Method 2. Total energy of the system is

$$E = \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 \quad \rightarrow \quad E = \frac{1}{2}\left(\frac{1}{12}M\ell^2\right)\omega^2 + \frac{1}{2}k\frac{\ell^2}{4}(\sin\theta)^2$$

Since total energy is constant, we have

$$\frac{dE}{dt} = \left(\frac{1}{12}M\ell^2\right)\omega\frac{d\omega}{dt} + k\frac{\ell^2}{4}\sin\theta\cos\theta\,\frac{d\theta}{dt} = 0$$

Since $\dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}\,, \qquad \dfrac{d\theta}{dt} = \omega$, and $\cos\theta \approx 1, \sin\theta \approx \theta$ for small angles, we have

$$\frac{dE}{dt} = 0 \quad \rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{3k}{M}\theta = 0 \quad \rightarrow \quad \omega = \sqrt{\frac{3k}{M}}$$